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Nevanlinna extremal measures for some orthogonal polynomials related to birth and death processes

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Abstract

We consider two classes of birth and death processes which lead to indeterminate Stieltjes moment problems, and we calculate the entire functions C and D from the corresponding Nevanlinna matrix. This permits a direct computation of a Nevanlinna extremal orthogonality measure for each of the processes.

Keywords: Orthogonal polynomials; Indeterminate moment problems; Birth and death processes

0. Introduction

In the Nevanlinna parametrization of the full set V of solutions to an indeterminate moment problem the parameter space is the one-point compactification $\mathscr{P}\cup\{\infty\}$ of the set $\mathscr{P}$ of Pick functions, cf. [1, 5]. When the parameter is restricted to $\mathbb{R}\cup\{\infty\}$, we obtain the family $(\nu_t)_{t\in\mathbb{R}\cup\{\infty\}}$ of Nevanlinna extremal solutions satisfying

$$\int\frac{\mathrm{d}\nu_t(x)}{z-x}=\frac{A(z)t-C(z)}{B(z)t-D(z)},\quad z\in\mathbb{C}\backslash\mathbb{R},\tag{0.1}$$

where A, B, C, D are certain entire functions forming the so-called Nevanlinna matrix associated with the moment problem. For the precise definition of these functions see below.

It is known that ν_t is a discrete measure

$$\nu_t=\sum_{\lambda\in\Lambda_t}m_\lambda\varepsilon_\lambda,\tag{0.2}$$

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where ε_λ denotes the Dirac measure at λ, $\Lambda_t = \{z \in \mathbb{C} \mid B(z)t - D(z) = 0\}$ for $t \in \mathbb{R}$, $\Lambda_\infty = \{z \in \mathbb{C} \mid B(z) = 0\}$ and

$$m_\lambda = \frac{A(\lambda)t - C(\lambda)}{B'(\lambda)t - D'(\lambda)} > 0 \quad \text{for } \lambda \in \Lambda_t. \tag{0.3}$$

The Nevanlinna extremal solutions are characterized by Riesz [16] as those measures $\mu \in V$ for which the polynomials are dense in $L^2(\mu)$.

Despite their fundamental character very few explicit examples of Nevanlinna extremal measures are known. This is connected with the fact that the functions A, B, C, D are not easily expressible in closed form in concrete examples. Furthermore, it is not easy to verify directly that the polynomials are dense in $L^2(\mu)$ with respect to a given discrete measure.

The first explicit example of a Nevanlinna extremal solution is as far as we know pointed out by Chihara, cf. [6, p. 483]. It is the orthogonality measure $\beta^{(a)}$ for the Al-Salam–Carlitz q-polynomials $V_n(x)$ when $0 < q < a \leqslant 1$, cf. [2], [7, p. 197]. The measure $\beta^{(a)}$ is concentrated in the points q^{-n}, $n \geqslant 0$. The corresponding moment problem is determinate with respect to $[1, \infty[$ in the sense that $\beta^{(a)}$ is the only solution to the moment problem concentrated on $[1, \infty[$. This is the clue to seeing that $\beta^{(a)}$ is Nevanlinna extremal. The "direct method" of calculating $\beta^{(a)}$ via the entire functions C, D from the Nevanlinna matrix does not seem to have been carried out but will be given in Section 5. It turns out that $\beta^{(a)}$ is Nevanlinna extremal when $0 < q < a$, $aq < 1$.

Moak [15] considered q-analogues of the Laguerre polynomials leading to indeterminate moment problems. He calculated the corresponding functions B and D and expressed them in terms of q-Bessel functions.

In a recent paper Chihara and Ismail [9] considered Al-Salam–Chihara polynomials which correspond to an indeterminate moment problem for certain choices of the parameters, in which case the Nevanlinna matrix is calculated.

In this paper we shall consider the indeterminate moment problem corresponding to the quartic birth and death process already considered by Valent [17, 18]. A one-parameter family $(\tau_a)_{a \in [-1,1]}$ of discrete solutions to the moment problem was given in [18], namely

$$\tau_a = \tfrac{1}{2}(1 + a)\frac{\pi}{K_0^2}\varepsilon_{x_0} + \sum_{k=1}^{\infty} \frac{2\pi^2}{K_0^2}(1 + a(-1)^k)\frac{k}{\sinh(k\pi)}\varepsilon_{x_k}, \tag{0.4}$$

where $x_k = (k\pi/K_0)^4$, $k = 0, 1, \ldots$ and K_0 is a certain constant, see (3.5). In the family $(\tau_a)_{a \in [-1,1]}$ only τ_1 and τ_{-1} are Nevanlinna extremal, and τ_1 corresponds to the parameter value $t = 0$, i.e. $\tau_1 = \nu_0$. We shall show this in Section 3, where we calculate C and D using generating functions determined in [13, 19]. The expression for D is particularly simple, since it can be expressed by elementary functions

$$D(z) = \frac{4}{\pi} z^{1/2} \sin\left(\frac{K_0}{2} z^{1/4}\right) \sinh\left(\frac{K_0}{2} z^{1/4}\right). \tag{0.5}$$

The moment problem is indeterminate in the sense of Stieltjes, so there is a certain parameter value $\alpha < 0$ such that the Nevanlinna extremal solutions $(\nu_t)_{t \in \mathbb{R} \cup \{\infty\}}$ are supported by $[0, \infty[$ precisely for $t \in [\alpha, 0]$. It is proved in [19] that $\tau_{-1} = \nu_\alpha$. In [4] also A and B are calculated, and we prove that α is given by

$$-\frac{1}{\alpha}=\frac{1}{4\sqrt{2}}\int_0^{K_0} t^2 \operatorname{cn} t \, \mathrm{d}t. \tag{0.6}$$

The formulas for A and B are given without proof in Section 4.

Sections 1 and 2 are of preparatory nature, where we put together mostly known results from the theory of the indeterminate moment problem and birth and death processes.

1. The Nevanlinna matrix

Let $s=(s_n)_{n\geqslant 0}$ be a normalized ($s_0=1$) Hamburger moment sequence, and let V denote the set of probabilities on $\mathbb{R}$ having s as sequence of moments. We write $\det(H)$ and $\operatorname{indet}(H)$ to distinguish if the moment problem is determinate (V consists of one measure) or indeterminate (V is an infinite set). If there is at least one measure in V supported by $[0,\infty[$ then s is a Stieltjes moment sequence. We use the notation $\det(S)$ or $\operatorname{indet}(S)$ to distinguish if there is precisely one measure in V which is supported by $[0,\infty[$ or if there are infinitely many. It is well known that a Stieltjes moment problem can be $\operatorname{indet}(H)$ and $\det(S)$.

In the rest of this section we assume the problem to be $\operatorname{indet}(H)$.

The functional $\mathscr{L}$ defined on the vector space $\mathscr{C}[x]$ of polynomials $p(x)=\sum_0^n c_k x^k$ by

$$\mathscr{L}(p)=\sum_{k=0}^{n} c_k s_k=\int p\,\mathrm{d}\mu$$

is independent of $\mu\in V$. The orthonormal polynomials $(P_n)_{n\geqslant 0}$ associated with s are uniquely determined by the requirement that P_n has positive leading coefficient for each n. The polynomials of the second kind are defined by

$$Q_n(x)=\mathscr{L}_y\left(\frac{P_n(x)-P_n(y)}{x-y}\right). \tag{1.1}$$

Since the problem is indeterminate $(P_n(z)),(Q_n(z))\in l_2$ and we define the Nevanlinna matrix

$$\begin{pmatrix} A & C\\ B & D\end{pmatrix} \tag{1.2}$$

of entire functions in the usual way, cf. [1]:

$$\begin{aligned} A(z)&=z\sum_{k=0}^{\infty}Q_k(0)Q_k(z), & C(z)&=1+z\sum_{k=0}^{\infty}P_k(0)Q_k(z),\\ B(z)&=-1+z\sum_{k=0}^{\infty}Q_k(0)P_k(z), & D(z)&=z\sum_{k=0}^{\infty}P_k(0)P_k(z). \end{aligned} \tag{1.3}$$

We recall that the function

$$\rho(x)=\left(\sum_{k=0}^{\infty}P_k(x)^2\right)^{-1},\quad x\in\mathbb{R} \tag{1.4}$$

plays an important role, cf. [1, pp. 63, 114]. First of all, the mass m_λ at $\lambda \in \Lambda_t$, for the Nevanlinna extremal measure ν_t is given as $m_\lambda = \rho(\lambda)$, and for all $\lambda \in \mathbb{R}$ we have

$$\rho(\lambda) = \sup_{\sigma \in V} \sigma(\{\lambda\}). \tag{1.5}$$

Secondly, if a measure $\nu \in V$ has maximal mass $\nu(\{\lambda\}) = \rho(\lambda)$ at some point $\lambda \in \mathbb{R}$, then ν is Nevanlinna extremal, and the parameter $t \in \mathbb{R} \cup \{\infty\}$ is given by $t = D(\lambda)/B(\lambda)$. Note that

$$\rho(0) = \frac{1}{D'(0)}. \tag{1.6}$$

2. Birth and death processes

A birth and death process is determined by the sequences $(\lambda_n)_{n \geqslant 0}$ of birth rates and $(\mu_n)_{n \geqslant 0}$ of death rates, cf. [12] where $\lambda_n > 0$, $\mu_{n+1} > 0$ for $n \geqslant 0$, $\mu_0 \geqslant 0$. We shall always assume $\mu_0 = 0$. For the solution of Kolmogorov's equations we consider the difference equation

$$(\lambda_n + \mu_n - x) F_n(x) = \mu_{n+1} F_{n+1}(x) + \lambda_{n-1} F_{n-1}(x), \quad n \geqslant 0, \tag{2.1}$$

with the initial conditions

$$F_{-1}(x) = 0, \qquad F_0(x) = 1, \tag{2.2}$$

which determines F_n as a polynomial of degree n for $n \geqslant 0$. Defining

$$\pi_0 = 1, \qquad \pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}, \quad n \geqslant 1 \tag{2.3}$$

and

$$a_n = \lambda_n + \mu_n, \quad b_n = \sqrt{\lambda_n \mu_{n+1}}, \qquad n \geqslant 0,$$

we see that the polynomials

$$P_n(x) = \frac{(-1)^n}{\sqrt{\pi_n}} F_n(x) \tag{2.4}$$

satisfy the recurrence relation

$$x P_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x) \tag{2.5}$$

with the initial conditions

$$P_{-1}(x) = 0, \qquad P_0(x) = 1. \tag{2.6}$$

By Favard's theorem $(P_n)_{n \geqslant 0}$ are the orthonormal polynomials with respect to a normalized Hamburger moment sequence s. It is known that s is a Stieltjes moment sequence.

Replacing (λ_n) and (μ_n) in (2.1) by the shifted birth and death rates $\tilde{\lambda}_n = \lambda_{n+1}$, $\tilde{\mu}_n = \mu_{n+1}$, $n \geqslant 0$, we let $\tilde{F}_n(x)$ denote the corresponding solution to (2.1) and (2.2). It is easy to see that

$$Q_n(x) = \frac{1}{\mu_1} \frac{(-1)^{n-1}}{\sqrt{\pi_n}} \tilde{F}_{n-1}(x), \quad n \geqslant 0. \tag{2.7}$$

In fact, denoting the right-hand side $\tilde{Q}_n(x)$, we see that $\tilde{Q}_n(x)$ satisfies the recursion (2.1) with the initial conditions

$$\tilde{Q}_0(x) = 0, \qquad \tilde{Q}_1(x) = \frac{1}{\mu_1 \sqrt{\pi_1}} = \frac{1}{b_0}, \tag{2.8}$$

which characterizes $Q_n(x)$, cf. [1, p. 8].

Proposition 2.1. *Suppose that the birth and death rates (λ_n) and (μ_n) (with $\mu_0 = 0$) lead to a problem which is* indet(H). *For $z \in \mathbb{C}$ we have*

$$C(z) = 1 - \frac{z}{\mu_1} \sum_{n=0}^{\infty} \tilde{F}_n(z), \qquad D(z) = z \sum_{n=0}^{\infty} F_n(z), \tag{2.9}$$

where the series converge uniformly on compact subsets of $\mathbb{C}$.

Proof. From (2.1) we get $F_n(0) = \pi_n$ and hence $P_n(0)P_n(z) = F_n(z)$ and $P_n(0)Q_n(z) = -(1/\mu_1)\tilde{F}_{n-1}(z)$, $n \geqslant 0$, so (2.9) follows from (1.3) □

In order to decide whether a given birth and death process leads to an indeterminate moment problem we shall use the criterion of Stieltjes given in the appendix to [1]. In order to do this we shall determine the coefficients (m_n) and (l_n) in the associated continued fraction. According to [1, p. 233] we have

$$b_n = \frac{1}{l_{n+1}\sqrt{m_{n+1}m_{n+2}}}, \quad n \geqslant 0,$$

$$a_n = \frac{1}{m_{n+1}}\left(\frac{1}{l_n} + \frac{1}{l_{n+1}}\right), \quad n \geqslant 1, \qquad a_0 = \frac{1}{m_1 l_1}.$$

Defining $m_1 = 1$ we find $l_1 = 1/a_0 = 1/\lambda_0$, and it is easy to see by induction that

$$\left.\begin{aligned} m_n &= \pi_{n-1} \\ l_n &= \frac{1}{\pi_n \mu_n} \end{aligned}\right\}, \quad n \geqslant 1. \tag{2.10}$$

We therefore have the following result.

Proposition 2.2. *The moment problem associated with the birth and death process with rates (λ_n) and (μ_n), $\mu_0 = 0$, is:*

(a) $\operatorname{indet}(S)$ *if and only if* $\sum_{n=0}^{\infty}\pi_n < \infty$ *and* $\sum_{n=1}^{\infty} 1/\pi_n\mu_n < \infty$,
(b) $\operatorname{indet}(H)$ *if and only if* $\sum_{n=1}^{\infty}\pi_n(\sum_{k=1}^{n} 1/\pi_k\mu_k)^2 < \infty$,
(c) $\det(S)$ *and* $\operatorname{indet}(H)$ *if and only if* $\sum_{n=1}^{\infty} 1/\pi_n\mu_n = \infty$ *and* $\sum_{n=1}^{\infty}\pi_n(\sum_{k=1}^{n} 1/\pi_k\mu_k)^2 < \infty$.

3. Determination of C and D in a case of quartic rates

We shall consider the rates introduced in [13, 18]:

$$\left.\begin{aligned}\lambda_n &= (4n+1)(4n+2)^2(4n+3)\\ \mu_n &= (4n-1)(4n)^2(4n+1)\end{aligned}\right\}, \quad n \geqslant 0. \tag{3.1}$$

In this case we find

$$\pi_n = \frac{1}{4n+1}\left(\frac{(\frac{1}{2})_n}{n!}\right)^2 \sim \frac{1}{4\pi}\frac{1}{n^2}. \tag{3.2}$$

Furthermore,

$$\frac{\lambda_k}{\mu_k} = \left(1+\frac{1}{2k}\right)^2\left(1+\frac{1}{k-\frac{1}{4}}\right), \quad k \geqslant 1,$$

so

$$\mu_{n+1}\pi_{n+1} = \lambda_0 \prod_{k=1}^{n}\frac{\lambda_k}{\mu_k} = 12\prod_{k=0}^{n-1}\left(1+\frac{1}{k+\frac{3}{4}}\right)\left(\prod_{k=0}^{n-1}\left(1+\frac{1}{2k+2}\right)\right)^2.$$

Using that

$$\prod_{k=0}^{n-1}\left(1+\frac{1}{ak+b}\right) = \frac{((b+1)/a)n}{(b/a)n} \sim \frac{\Gamma(b/a)}{\Gamma((b+1)/a)}n^{1/a}, \quad a, b > 0,$$

we find

$$\mu_{n+1}\pi_{n+1} \sim \frac{64}{\pi}n^2. \tag{3.3}$$

It follows from Proposition 2.2(a) that the corresponding moment problem is $\operatorname{indet}(S)$.

We shall next determine the entire functions C and D. For this we use the entire functions

$$\delta_l(z) = \sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+1}}{(4n+l)!}, \quad l = 0, 1, 2, 3 \tag{3.4}$$

called trigonometric of order 4 in [10] and used heavily in [18].

We shall also use the elliptic functions in the lemniscatic case corresponding to the modulus $k = 1/\sqrt{2}$. We therefore omit k from the notation and follow [20]. The corresponding complete elliptic integral $K_0 = K(1/\sqrt{2})$ is given by

$$\frac{1}{\sqrt{2}}K_0 = \int_0^1 \frac{du}{\sqrt{1-u^4}} = \frac{\Gamma(\frac{1}{4})^2}{4\sqrt{2\pi}}. \tag{3.5}$$

We further define the entire functions

$$\Delta_l(z) = \frac{K_0}{\sqrt{2}} \int_0^1 \delta_l(uz)\,\mathrm{cn}\,(K_0 u)\,\mathrm{d}u, \quad l = 0, 1, 2, 3. \tag{3.6}$$

Theorem 3.1. *The entire functions C and D from the Nevanlinna matrix are given as*

$$C(z) = \frac{4}{\pi} \Delta_0\left(\frac{z^{1/4} K_0}{\sqrt{2}}\right), \qquad D(z) = \frac{4}{\pi} \sqrt{z}\,\delta_2\left(\frac{z^{1/4} K_0}{\sqrt{2}}\right). \tag{3.7}$$

Proof. The generating function for $(F_n(z))$ has been determined in [13] and reads ($z \in \mathbb{C}, 0 < w \leqslant 1$)

$$\sum_{n=0}^{\infty} F_n(z) w^n = \frac{1}{\pi} \int_0^1 \frac{\delta_l(z^{1/4}\theta((uw)^{1/4}))}{(zuw)^{1/4}} \frac{\mathrm{d}u}{\sqrt{u(1-u)}}, \tag{3.8}$$

where

$$\theta(w) = \int_0^w \frac{\mathrm{d}u}{\sqrt{1-u^4}}, \tag{3.9}$$

and the inverse function is

$$w = t(\theta) = \frac{1}{\sqrt{2}} \mathrm{sd}(\sqrt{2}\theta) = \frac{1}{\sqrt{2}} \frac{\mathrm{sn}(\sqrt{2}\theta)}{\mathrm{dn}(\sqrt{2}\theta)}. \tag{3.10}$$

For $w = 1$ we get

$$D(z) = \frac{z}{\pi} \int_0^1 \frac{\delta_l(z^{1/4}\theta(u^{1/4}))}{(zu)^{1/4}} \frac{\mathrm{d}u}{\sqrt{u(1-u)}},$$

and the change of variable

$$u^{1/4} = \frac{1}{\sqrt{2}} \mathrm{sd}(\sqrt{2}\theta) \tag{3.11}$$

for which

$$\frac{\mathrm{d}u}{\sqrt{u(1-u)}} = 4u^{1/4}\,\mathrm{d}\theta$$

leads to

$$D(z) = \frac{4}{\pi} z \int_0^{K_0/\sqrt{2}} \frac{\delta_1(z^{1/4}\theta)}{z^{1/4}} \mathrm{d}\theta = \frac{4}{\pi} z^{1/2} \delta_2\left(z^{1/4} \frac{K_0}{\sqrt{2}}\right)$$

because $\delta_2' = \delta_1$.

To determine $C(z)$ we use the following formula involving $(\tilde{F}_n(z))$, $(z \in \mathbb{C},\ 0 \leqslant w \leqslant 1)$

$$\tilde{F}_2(z, w) := \frac{1}{\mu_1} \sum_{n=0}^{\infty} \frac{(n+1)!}{(\frac{1}{2})_{n+1}} \tilde{F}_n(z) w^{4n+5}$$
$$= \frac{1}{\sqrt{2}} \int_0^{\theta(w)} \frac{\delta_3(z^{1/4}(\theta(w) - u))}{z^{3/4}} \operatorname{sd}(\sqrt{2}u)\, du. \tag{3.12}$$

To derive this we specialize (54) in [19] to $\mu = 0$, $c = 1$ and integrate by parts twice using the identity

$$2t(\theta)^5 = t(\theta) - \frac{d^2}{d\theta^2}\left(\frac{t(\theta)^3}{3!}\right), \tag{3.13}$$

where $t(\theta)$ is given by (3.10). (Notice that δ_l here differs from δ_l in [19] by a factor $\exp(il\pi/4)$.)

From (3.12) we can check that

$$\tilde{F}_1(z, w) := \frac{1}{\mu_1} \sum_{n=0}^{\infty} \tilde{F}_n(z) w^{n+1} = \frac{1}{\pi} \int_0^1 \frac{\tilde{F}_2(z, (wu)^{1/4})}{(wu)^{1/4}} \frac{du}{\sqrt{u(1-u)}},$$

just by expanding the right-hand member in powers of u and integrating termwise.

It follows by Proposition 2.1 that

$$1 - C(z) = \frac{z}{\pi} \int_0^1 \frac{\tilde{F}_2(z, u^{1/4})}{u^{1/4}} \frac{du}{\sqrt{u(1-u)}}.$$

Using again the change of variable (3.11) gives

$$\tilde{F}_2(z, u^{1/4}) = \frac{1}{\sqrt{2}} \int_0^{\theta} \frac{\delta_3(z^{1/4}(\theta - \tau))}{z^{3/4}} \operatorname{sd}(\sqrt{2}\tau)\, d\tau,$$

so that

$$1 - C(z) = \frac{4z}{\pi} \int_0^{K_0/\sqrt{2}} \tilde{F}_2(z, u^{1/4})\, d\theta$$
$$= \frac{4z^{1/4}}{\pi} \int_0^{K_0/\sqrt{2}} \left(\frac{1}{\sqrt{2}} \int_0^{\theta} \delta_3(z^{1/4}(\theta - \tau)) \operatorname{sd}(\sqrt{2}\tau)\, d\tau\right) d\theta.$$

Interchanging the order of integration and using $\delta_0' = -\delta_3$ we get

$$1 - C(z) = \frac{4}{\pi} \int_0^{K_0/\sqrt{2}} \frac{\operatorname{sd}(\sqrt{2}\tau)}{\sqrt{2}} \left(1 - \delta_0\left(z^{1/4}\left(\frac{K_0}{\sqrt{2}} - \tau\right)\right)\right) d\tau$$
$$= \frac{4}{\pi} \int_0^{K_0} \frac{\operatorname{cn}(v)}{\sqrt{2}} \left(1 - \delta_0\left(\frac{1}{\sqrt{2}} z^{1/4} v\right)\right) dv,$$

where we have used the substitution $v = K_0 - \sqrt{2}\tau$.

Since

$$\int_0^{K_0} \operatorname{cn} v \, dv = \frac{\pi\sqrt{2}}{4},$$

we finally get

$$C(z) = \frac{2\sqrt{2}}{\pi} \int_0^{K_0} \delta_0\left(\frac{z^{1/4}}{\sqrt{2}} u\right) \operatorname{cn} u \, du = \frac{4}{\pi} \Delta_0\left(\frac{z^{1/4} K_0}{\sqrt{2}}\right). \qquad \square$$

Remark 3.2. It is clear from the definition of δ_l that C and D are entire functions as they should be. It is easy to see that $\delta_2(\sqrt{2}z) = \sin z \sinh z$ which gives the simple expression (0.5) for D.

Proposition 3.3. *The entire function D has order $\frac{1}{4}$ and type $K_0/\sqrt{2}$.*

Proof. For an entire function $f(z) = \sum_0^\infty a_n z^n$ the *order* ρ is given as

$$\rho = \limsup_{n\to\infty} \frac{n \log n}{-\log |a_n|}, \tag{3.14}$$

and if $0 < \rho < \infty$ then the *type* σ is given as

$$\sigma = \frac{1}{\mathrm{e}\rho} \limsup_{n\to\infty} n |a_n|^{\rho/n}, \tag{3.15}$$

cf. [14].

For

$$D(z) = \sum_{k=0}^{\infty} d_k z^k$$

we find

$$d_k = \frac{4}{\pi} (-1)^{k-1} \frac{(K_0/\sqrt{2})^{4k-2}}{(4k-2)!}, \quad k \geqslant 1, \tag{3.16}$$

and from Stirling's formula we get

$$\rho = \frac{1}{4}, \quad \sigma = \frac{K_0}{\sqrt{2}}. \qquad \square$$

Remark 3.4. In the same way one can determine the order and type of C to be 1/4 and $K_0/\sqrt{2}$, respectively. The fact that C and D have same order and same type is no coincidence, since it has recently been proved in [3] that all four functions in the Nevanlinna matrix associated with an indeterminate moment problem have the same order and the same type.

Using the formulas for C and D one can calculate the measure ν_0 given by (0.2) and see that it coincides with the measure τ_1 from (0.4).

Proposition 3.5. *The Nevanlinna extremal measure* ν_0 *is given as*

$$\nu_0 = \frac{\pi}{K_0^2}\varepsilon_{x_0} + \frac{4\pi^2}{K_0^2}\sum_{k=1}^{\infty}\frac{2k}{\sinh(2k\pi)}\varepsilon_{x_k}, \tag{3.17}$$

where

$$x_k = ((2k\pi)/K_0)^4, \quad k = 0, 1, \ldots . \tag{3.18}$$

Proof. From (0.5) it is clear that the zeros of D are given by (3.18). We shall verify that

$$\frac{C(0)}{D'(0)} = \frac{\pi}{K_0^2}, \qquad \frac{C(x_k)}{D'(x_k)} = \frac{8\pi^2 k}{K_0^2\sinh(2k\pi)} \quad \text{for } k \geqslant 1. \tag{3.19}$$

It is easy to see that

$$D'(0) = \frac{K_0^2}{\pi}, \qquad D'(x_k) = (-1)^k K_0^2\frac{\sinh(k\pi)}{4\pi^2 k}, \quad k \geqslant 1.$$

Using the formula for I_0 in the appendix to [19] one gets

$$C(x_k) = \frac{(-1)^k}{\cosh(k\pi)}, \quad k \geqslant 0,$$

and (3.19) follows. □

Remark 3.6. Without calculating $C(x_k)$ it is still possible to argue that τ_1 from (0.4) is equal to ν_0. In fact, since $\tau_1 \in V$ and

$$\tau_1(\{0\}) = \frac{\pi}{K_0^2} = \frac{1}{D'(0)},$$

this follows by (1.6).

4. On the entire functions A, B associated with quartic rates

It is difficult to make a direct computation of the functions A and B mainly because there is no closed form for $Q_n(0)$. Using asymptotic analysis of $P_n(z)$ and $Q_n(z)$ one can get formulas for A and B. The leading terms in the asymptotics are not sufficient to obtain A and B; one has to include subleading terms. The details of this analysis are given in [4]. The final result is

$$A(z) = \frac{1}{\sqrt{z}}\Delta_2\left(\frac{z^{1/4}K_0}{\sqrt{2}}\right) - \frac{4}{\pi}\xi\Delta_0\left(\frac{z^{1/4}K_0}{\sqrt{2}}\right), \tag{4.1}$$

$$B(z) = -\delta_0\left(\frac{z^{1/4}K_0}{\sqrt{2}}\right) - \frac{4}{\pi}\xi\sqrt{z}\delta_2\left(\frac{z^{1/4}K_0}{\sqrt{2}}\right), \tag{4.2}$$

where

$$\xi = \lim_{z\to 0} z^{-1/2} \Delta_2\left(\frac{z^{1/4}K_0}{\sqrt{2}}\right) = \frac{1}{4\sqrt{2}}\int_0^{K_0} t^2 \operatorname{cn} t\, dt. \tag{4.3}$$

Note that by (0.6) $\xi = -1/\alpha$.

5. Al-Salam–Carlitz q-polynomials

These are the monic polynomials $V_n = V_n^{(a)}(x; q)$ introduced in [2] and determined by the recursion

$$V_{n+1}(x) = (x - (1 + a)q^{-n})V_n(x) - aq^{-(2n-1)}(1 - q^n)V_{n-1}(x) \tag{5.1}$$

and the initial conditions

$$V_{-1}(x) = 0, \qquad V_0(x) = 1. \tag{5.2}$$

We shall always restrict the parameters a, q to the domain $a > 0$, $0 < q < 1$, and use the notation

$$(z; q)_n = \prod_{j=1}^{n} (1 - zq^{j-1}), \quad z \in \mathbb{C},\ n = 0, 1, \ldots, \infty \tag{5.3}$$

(with $(z; q)_0 = 1$).

Defining

$$\sigma_n = (-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n} \tag{5.4}$$

and

$$F_n(x) = \sigma_n V_n(x + 1), \tag{5.5}$$

we see that (5.1) is transformed into (2.1) with the birth and death rates

$$\lambda_n = aq^{-n}, \quad \mu_n = q^{-n} - 1, \qquad n \geqslant 0, \tag{5.6}$$

i.e.

$$((1 + a)q^{-n} - 1 - x)F_n(x) = (q^{-n-1} - 1)F_{n+1}(x) + aq^{-n+1}F_{n-1}(x). \tag{5.7}$$

We find, cf. (2.3) and (2.10)

$$m_{n+1} = \pi_n = \frac{(aq)^n}{(q; q)_n}, \qquad l_n = \frac{1}{\mu_n \pi_n} = \frac{(q;q)_{n-1}}{a^n}, \tag{5.8}$$

which by Proposition 2.2 lead to the following conclusions already found in [8]: within the domain $a > 0$, $0 < q < 1$ the problem is

$$\begin{aligned} &\operatorname{indet}(S) \Leftrightarrow 1 < a < q^{-1},\\ &\det(S), \operatorname{indet}(H) \Leftrightarrow q < a \leqslant 1,\\ &\det(H) \Leftrightarrow q \geqslant a \text{ or } aq \geqslant 1. \end{aligned} \tag{5.9}$$

The following generating function involving V_n was given in [2], cf. [7, p. 195]

$$\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n-1)/2}}{(q;q)_n}V_n(x)w^n=\frac{(wx;q)_\infty}{(w;q)_\infty(aw;q)_\infty},\tag{5.10}$$

which by (5.4), (5.5) leads to

$$\sum_{n=0}^{\infty}F_n(z)w^n=\frac{(w(1+z)q;q)_\infty}{(wq;q)_\infty(awq;q)_\infty}.\tag{5.11}$$

Without referring to earlier work this formula can be easily established. In fact, if $F(z, w)$ denotes the left-hand side of (5.11), multiplication with w^n in (5.7) and summation leads to

$$F\left(z,\frac{w}{q}\right)\left\{1+a-\frac{1}{w}-aw\right\}=F(z,w)\left\{1+z-\frac{1}{w}\right\}\tag{5.12}$$

or equivalently, replacing w by wq

$$F(z,w)=\frac{1-(1+z)wq}{(1-wq)(1-awq)}F(z,wq).\tag{5.13}$$

Iterating this we get (5.11) provided that for each $z\in\mathbb{C}$ the series on the left-hand side of (5.11) converges for $|w|$ sufficiently small. We shall restrict the attention to the case, where the problem is indet(H), because we then know by Proposition 2.1 that for each $z\in\mathbb{C}$ the left-hand side of (5.11) converges uniformly for $|w|\leqslant 1$.

We therefore have

(5.11) *holds for* $z\in\mathbb{C}$, $|w|\leqslant 1$ *when the problem is* indet(H).

Let $\tilde{F}_n(z)$ solve the recurrence relation

$$((1+a)q^{-n-1}-1-x)\tilde{F}_n(x)=(q^{-n-2}-1)\tilde{F}_{n+1}(x)+aq^{-n}\tilde{F}_{n-1}(x)\tag{5.14}$$

with $\tilde{F}_{-1}\equiv 0$, $\tilde{F}_0\equiv 1$. We shall determine the generating function

$$\tilde{F}(z,w)=\sum_{n=0}^{\infty}\tilde{F}_n(z)w^n.\tag{5.15}$$

Multiplying (5.14) with w^n and summing leads to

$$\frac{1+a-w^{-1}-aw}{q}\tilde{F}\left(z,\frac{w}{q}\right)=\left(1+z-\frac{1}{w}\right)\tilde{F}(z,w)+\frac{1}{w}-\frac{1}{qw}$$

or equivalently, replacing w by wq

$$\tilde{F}(z,w)=q\frac{1-w(1+z)q}{(1-wq)(1-awq)}\tilde{F}(z,wq)+\frac{1-q}{(1-wq)(1-awq)}.\tag{5.16}$$

Iterating and using $\tilde{F}(z,wq^n)\to 1$ for $n\to\infty$ gives the following result.

Proposition 5.1. *Suppose the problem is* indet(H). *The generating function* (5.15) *is given by*

$$\tilde{F}(z,w)=(1-q)\sum_{n=0}^{\infty}\frac{(w(1+z)q;q)_n}{(wq;q)_{n+1}(awq;q)_{n+1}}q^n,\quad z\in\mathbb{C},\ |w|\leqslant 1.\tag{5.17}$$

We shall now determine the functions C and D when the problem is indet(H), which by (5.9) corresponds to $q < a$ and $aq < 1$. Using Proposition 2.1 we get the following theorem.

Theorem 5.2. *Suppose $0 < q < a$, $aq < 1$. Then the entire functions C and D are given for $z \in \mathbb{C}$ by*

$$C(z) = \sum_{n=0}^{\infty} \frac{(1+z;q)_n}{(aq;q)_n(q;q)_n} q^n = {}_2\phi_1(1+z, 0; aq; q, q), \tag{5.18}$$

$$D(z) = z\frac{((1+z)q;q)_\infty}{(q;q)_\infty(aq;q)_\infty} = -\frac{(1+z;q)_\infty}{(q;q)_\infty(aq;q)_\infty}. \tag{5.19}$$

Remark 5.3. The zeros of D are $z_n = q^{-n} - 1$, $n \geqslant 0$. Since the convergence exponent of the zeros is 0, viz.

$$\sum_{n=1}^{\infty} \frac{1}{z_n^{\varepsilon}} < \infty$$

for every $\varepsilon > 0$, then D is of order zero. It follows by [3] that all four functions A, B, C, D from the Nevanlinna matrix are of order zero.

Using formula (0.3) we shall now determine the Nevanlinna extremal measure ν_0. It turns out that $\nu_0 = \varepsilon_{-1} * \beta^{(a)}/K$, i.e. the Al-Salam–Carlitz measure $\beta^{(a)}/K$ shifted by -1, cf. [7, p. 197]. Previously $\beta^{(a)}/K$ was known to be Nevanlinna extremal only in the case $0 < q < a \leqslant 1$, where the moment problem is indet(H) but determinate with respect to $[1, \infty[$.

Theorem 5.4. *Suppose $0 < q < a$, $aq < 1$. The measure ν_0 is given by*

$$\nu_0 = \sum_{n=0}^{\infty} m_n \varepsilon_{z_n},$$

where

$$z_n = q^{-n} - 1, \quad m_n = (aq;q)_\infty \frac{a^n q^{n^2}}{(q;q)_n(aq;q)_n}, \qquad n \geqslant 0. \tag{5.20}$$

Proof. We know that

$$m_n = \frac{C(z_n)}{D'(z_n)}, \quad n \geqslant 0,$$

and it is easy to see that

$$D'(z_n) = \frac{(-1)^n(q;q)_n}{(aq;q)_\infty} q^{-n(n-1)/2}. \tag{5.21}$$

From (5.18) we get

$$C(z_n) = \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(aq;q)_k(q;q)_k} q^k$$
$$= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{-kn} \frac{q^{k(k+1)/2}}{(aq;q)_k}, \tag{5.22}$$

where we use the Gaussian binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

We claim that (5.22) can be reduced to

$$C(z_n) = (-1)^n \frac{a^n q^{n(n+1)/2}}{(aq;q)_n}, \tag{5.23}$$

which together with (5.21) give (5.20).

Proof of (5.23). We proceed by induction using the formula

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} q^k + \begin{bmatrix} n \\ k-1 \end{bmatrix}. \tag{5.24}$$

Calling $S_n(a)$ the expression of (5.22), we have by (5.24)

$$S_{n+1}(a) = S_n(a) - \frac{q^{-n}}{1-aq} S_n(aq),$$

which permits the induction step, if (5.23) holds for some n and $a \neq q^{-j}, j \geqslant 1$. □

Remark 5.5. Since ν_0 is a probability measure we have

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n(aq;q)_n} = \frac{1}{(aq;q)_\infty}, \tag{5.25}$$

thereby evaluating the constant K in [7, p. 197]. Ismail has given a direct proof of this result making use of the Jacobi triple product identity, see Theorem 5.1 of [11].

Remark 5.6. The easiest way of seeing that $\varepsilon_{-1} * \beta^{(a)}/K$ is Nevanlinna extremal in the case $0 < q < a$, $aq < 1$ is to proceed like this: We know from [2, 7] that $\varepsilon_{-1} * \beta^{(a)}/K \in V$, and the mass at zero is $1/K$ which by [11] is $(aq;q)_\infty$. But this is the maximal mass since $\rho(0) = 1/D'(0) = (aq;q)_\infty$, cf. (1.6) and (5.21).

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